JOURNAL OF FUNCTIONAL ANALYSIS 76, 126–139 (1988)

Locally Compact Group Actions on C^*-Algebras and Compact Subgroups

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Communicated by A. Connes

Received September 16, 1985; revised August 10, 1986

Let (A, G, α) be a C^*-dynamical system and $K \subset G$ a compact subgroup. In this paper we give necessary and sufficient conditions in order that the crossed product $G \times_\alpha A$ be simple. Our conditions are in terms of the dual $\hat{K}$ of the compact subgroup K. We also consider the similar problem for prime crossed products. © 1988 Academic Press.

1. INTRODUCTION

Let (A, G, α) be a C^*-dynamical system with A a C^*-algebra, G a locally compact group, and $\alpha: G \to \mathrm{Aut}(A)$ a homomorphism such that the mapping $g \to \alpha_g(a)$ is continuous from G to A for every $a \in A$. We denote by A^α the fixed point algebra of α.

Let also $K \subset G$ be a compact subgroup. In this paper we shall give necessary and sufficient conditions in order that the crossed product $G \times_\alpha A$ be simple or prime. Our conditions are in terms of the dual $\hat{K}$ of the compact subgroup K. The paper is organized as follows.

In the second section we summarize some preliminary definitions and results about continuous Banach representations of compact groups. The same section contains the definitions of the subspaces of spherical functions $S_{\pi_1, \pi_2}(\pi_1, \pi_2 \in \hat{G})$, those of the algebras S_π, I_π, and the relations between these subspaces.

Section 3 contains a study of saturated actions of G on A in terms of the ideals $S_{\pi,\iota} * S_{\iota,\pi}$. We mention that for compact groups the notion of saturated action has been defined by Rieffel (see [19, Chap. 7]).

The same section contains conditions in order that the crossed product be simple. For compact abelian groups G, our result reduces to the following: $G \times_\alpha A$ is simple if and only if (a) $\mathrm{Sp}(\alpha) = \hat{G}$ (here $\mathrm{Sp}(\alpha)$ stands for the Arveson spectrum of α) and (b) A^α is simple.

It is then shown that similar results hold for prime C^*-crossed products.

0022-1236/88 $3.00

2. Subspaces of Spherical Functions of the Crossed Product

I. *Preliminaries on Banach Representations of Compact Groups*

Let K be a compact group. We shall denote by $\hat{K}$ the dual of K, i.e., the set of all unitary equivalence classes of irreducible representations of K. For each $\pi \in \hat{K}$ we denote also by π a representative of that class and by H_π the (finite-dimensional) Hilbert space of π. We let ι be the trivial one-dimensional representation of K.

If $\pi \in \hat{K}$, let χ_π be its normalized character $\chi_\pi(k) = d(\pi)\,\mathrm{tr}(\pi_k^{-1})$, where $d(\pi)$ is the dimension of H_π.

Let B be a Banach space and β a continuous representation of K on B. Associated with a $\pi \in \hat{K}$ are the following continuous operators on B:

$$P^\beta(\pi)\colon a \to \int_k \chi_\pi(k)\,\beta_k(a)\,dk$$

and

$$P^\beta_{ij}(\pi)\colon a \to d(\pi) \int_k \overline{\pi_{ji}(k)}\,\beta_k(a)\,dk, \qquad 1 \leqslant i, j \leqslant d(\pi),$$

where $[\pi_{ij}(k)]$ is the matrix of π_k in a fixed orthonormal basis of H_π.

For standard properties of these operators we refer the reader to [11, 12]. Here we note some of them for further use:

2.1. *Remarks.* (i) $P^\beta(\pi)\,P^\beta_{ij}(\pi) = P^\beta_{ij}(\pi)\,P^\beta(\pi) = P^\beta_{ij}(\pi)$.

(ii) If B is a C^*-algebra and β_k are *-automorphisms then $P^\beta(\bar{\pi})(B) = [P^\beta(\pi)(B)]^*$; where $\bar{\pi}$ is the conjugate representation of π.

(iii) $\beta_k(P_{ij}(\pi)(a)) = \sum_l \pi_{lj}(k)\,P^\beta_{il}(a)$.

If we denote by $[P^\beta_{ij}(\pi)(a)]$ the matrix in $B \otimes B(H\pi)$ with entries $P^\beta_{ij}(\pi)(a)$, then (iii) may be written:

(iv) $(\beta_k \otimes \iota)([P^\beta_{ij}(\pi)(a)]) = [P^\beta_{ij}(\pi)(a)] \cdot (1_B \otimes \pi_k)$.

We now make the following notations:

$$B_1(\pi) = \{a \in B \mid P^\beta(\pi)(a) = a\}$$

and

$$B_2(\pi) = \{[a_{ij}] \in B \otimes B(H_\pi) \mid (\beta_k \otimes \iota)([a_{ij}]) = [a_{ij}] \cdot (1_B \otimes \pi_k),\ k \in K\}.$$

Clearly $B_1(\pi) \subset B$ and $B_2(\pi) \subset B \otimes B(H_\pi)$, $\pi \in \hat{K}$. Using the above Remarks we also have

$$B_1(\pi) = \{P^\beta(\pi)(a) \mid a \in B\}$$

and

$$B_2(\pi) = \{[a_{ij}] \in B \otimes B(H_\pi) \mid \beta_k(a_{ij}) = \sum_l \pi_{lj}(k)\, a_{il}\}.$$

By Remark (iii) above $[P_{ij}^\beta(\pi)(a)] \in B_2(\pi)$ for every $a \in B$, $\pi \in \hat{K}$. The next lemma shows that every element of $B_2(\pi)$ is of this form.

2.2. LEMMA. $B_2(\pi) = \{[P_{ij}^\beta(\pi)(a)] \mid a \in B\}$.

Proof. The above discussion shows that $\{[P_{ij}^\beta(\pi)(a)] \mid a \in B\} \subset B_2(\pi)$. Conversely let $[a_{ij}] \in B_2(\pi)$. Denote $a = \sum_i a_{ii}$. Then a standard calculation using orthogonality relations shows that $P_{ij}^\beta(\pi)(a) = a_{ij}$.

The proof of the following lemma is similar to the one given in [12] for Hilbert spaces.

2.3. LEMMA. *$\sum_{\pi \in \hat{K}} B_1(\pi)$ is dense in B.*

2.4. LEMMA (see, for instance, [23, Lemma 4.4.2.4]). *Let $a \in B$ and suppose that $P^\beta(\pi)(a) = 0$ for all $\pi \in \hat{K}$. Then $a = 0$.*

II. *Spaces of Spherical Functions inside the Crossed Product*

Let (A, G, α) be a C^*-dynamical system. We denote by $C_c(G)$ the set of all continuous, compactly supported functions from G to A. Let $K \subset G$ be a compact subgroup. It is known that $L^1(K)$ can be imbedded in the multiplier algebra $M(G \times_\alpha A)$ of the crossed product.

If $\phi \in L^1(K)$ and $y \in C_c(G, A)$ then

$$(\phi * y)(g) = \int_K \phi(k)\, \alpha_k(y(k^{-1}g))\, dk$$

and

$$(y * \phi)(g) = \int_K \phi(k)\, f(gk^{-1})\, dk.$$

In particular, if $\phi = \chi_\pi$ then ϕ is a projection in $M(G \times_\alpha A)$.

Let us define the following representation τ of K on the (Banach space) $C_c(G, A)$:

$$(\tau_k\, y)(g) = \alpha_k(y(k^{-1}g)).$$

Then τ extends to a continuous representation of K on (the Banach space) $G \times_\alpha A$.

2.5. LEMMA. (i) *$P^\tau(\pi)f = \chi_\pi * f$ for $\pi \in \hat{K}$, $f \in G \times_\alpha A$.*

(ii) *If $\chi_\pi * f = 0$ for all $\pi \in \hat{K}$, then $f = 0$.*

(iii) *$\sum_{\pi \in \hat{K}} \chi_\pi * (G \times A)$ is dense in $G \times_\alpha A$.*

(iv) *If $\pi_1 \neq \pi_2$ in $\hat{K}$ then the projections χ_{π_1} and χ_{π_2} of $M(G \times_\alpha A)$ are orthogonal.*

(v) *$\sum_{\pi \in \hat{K}} \chi_\pi = 1$ (as projections in $M(G \times_\alpha A)$).*

Proof. (i) follows from definitions, (ii) is a consequence of (i) and Lemma 2.4, (iii) follows from (i) and Lemma 2.3, (iv) is an easy consequence of orthogonality relations, and (v) follows from (iv) and (iii).

Let us denote $S_{\pi_1,\pi_2} = \chi_{\pi_1} * (G \times_\alpha A) * \chi_{\pi_2} (\pi_1, \pi_2 \in \hat{K})$. Then S_{π_1,π_2} are closed subspaces of $G \times_\alpha A$.

If $\pi_1 = \pi_2 = \pi$ we put $S_\pi = S_{\pi,\pi}$. Then S_π is a hereditary C^*-subalgebra of $G \times_\alpha A$ (that is, the set of positive elements of S_π is a hereditary subcone of the cone of positive elements in $G \times_\alpha A$; see [18] for definitions and properties of hereditary subalgebras).

2.6. LEMMA. (i) *If π_1, π_2, π_3, $\pi_4 \in \hat{K}$ and $\pi_2 \neq \pi_3$ then $S_{\pi_1,\pi_2} * S_{\pi_3,\pi_4} = 0$.*

(ii) $S_{\pi_1,\pi_2} * S_{\pi_2,\pi_3} = S_{\pi_1,\pi_3}$.

(iii) $(S_{\pi_1,\pi_2})^* = S_{\pi_2,\pi_1}$.

(iv) *$S_{\pi_1,\pi_2} * S_{\pi_2,\pi_1}$ is a two-sided ideal of S_{π_1}.*

(v) *Assume that $G = K$ is compact. For each $a \in A$ denote by $\tilde{a} \in C_c(G, A)$ the constant function $\tilde{a}(g) = a$. In this case $S_{\pi,\iota} = \{\tilde{a} \mid a \in A_1(\pi)\}$.*

Proof. Part (i) follows from Lemma 2.5(iv). Parts (ii), (iii), (iv), and (v) follow from definitions.

From now on we shall assume for simplicity that the group G is unimodular. This assumption is not essential but we make it in order to simplify our computations.

We now define the algebra I generated by the K-central functions in $G \times_\alpha A$. A function $y \in C_c(G, A)$ is called K-central if

$$\alpha_k(y(k^{-1}gk)) = y(g) \qquad \text{for all } k \in K,\ g \in G$$

(see, for example, [23, Sect. 4.5]). Then I is a C^*-subalgebra of $G \times_\alpha A$. We make the following notations:

$$I_\pi = I \cap S_\pi$$

$$I(\pi) = C(K) * \chi_\pi.$$

Then $I(\pi)$ is the full $d(\bar{\pi}) \times d(\bar{\pi})$ matrix algebra. The following proposition is the variant of [23, Proposition 4.5.1.8] in our setting.

2.7. PROPOSITION. *The bilinear map* $(f, y) \to f * y$ *from* $I(\pi) \times I_\pi$ *into* $G \times_\alpha A$ *when lifted to a linear map of* $I(\pi) \times I_\pi$ *into* $G \times A$ *establishes a* $*$*-algebra isomorphism of the tensor product* $I(\pi) \times I_\pi$ *with* S_π.

For the theory of imprimitivity bimodules and Morita equivalence of C^*-algebras we refer the reader to [20, 21].

We note the following corollary of the above proposition:

2.8. COROLLARY. I_π *is strongly Morita equivalent with* S_π. *Therefore*

(i) I_π *is simple if and only if* S_π *is simple,*

(ii) I_π *is prime if and only if* S_π *is prime,*

(iii) I_π *is type I if and only if* S_π *is type I.*

Proof. I_π is strongly Morita equivalent with S_π by Proposition 2.7(i) and (ii) follows from [21, Theorem 3.1] and (iii) follows from [20, Theorem 6.23].

We consider the following map P from $C_c(G, A)$ into itself:

$$(Py)(g) = \int_K \alpha_k(y(k^{-1}gk))\, dk. \tag{1}$$

It is then easily seen that $P(C_c(G, A)) \subset I$, and P can be extended to a projection of $G \times_\alpha A$ onto I. Another fact that will be used is the following observation: If $f \in I$ and $\chi_\pi * f = f$, then also $f = f * \chi_\pi$. Thus, the map $f \to \chi_\pi * f$ is a projection of I onto I_π.

Let us now specialize the above notions to the case when $G = K$. The following result is due to Landstad.

2.9. LEMMA [14, Lemma 3]. *Define a linear map* $\phi\colon C_c(G, A) \to A \otimes B(H_\pi)$ *by* $\phi(f) = \int F(g^{-1}) \otimes \overline{\pi_g}\, dg$. *Then* ϕ *is a* $*$*-isomorphism of* I_π *onto the fixed point algebra* $A \otimes B(H_{\bar{\pi}})^{\alpha \otimes ad\bar{\pi}}$. *The inverse of this isomorphism is the map* ψ *from* $A \otimes B(H_{\bar{\pi}})^{\alpha \otimes Ad\pi}$ *defined by*

$$\psi(b)(g) = d(\bar{\pi})\, \mathrm{tr}(b(1 \otimes \bar{\pi}(g))).$$

From the above lemma and its proof it follows that I_π consists only of continuous functions.

2.10. LEMMA. $P(\overline{S_{\pi,\iota} * S_{\iota,\pi}}) = \psi(\overline{A_2(\bar{\pi}) * A_2(\bar{\pi})})$.

Proof. By Lemma 2.6(v), $S_{\pi,\iota}$ is the closure of the set of all $\tilde{a}$ with $a \in A_1(\pi)$. By Lemma 2.6(iii), $S_{\iota,\pi}$ is the closure of all $(\tilde{b})^*$ with $b \in A_1(\pi)$. Since, by definition of the involution in $G \times_\alpha A$, $(\tilde{b})^*(g) = \alpha_g(b^*)$, it follows that $S_{\pi,\iota} * \overline{S_{\iota,\pi}}$ is generated by all functions of the form

$$\tilde{a} * (\tilde{b})^*(g) = a\alpha_g(b^*), \qquad a, b \in A_1(\pi).$$

We have

$$\begin{aligned} P(\tilde{a} * (\tilde{b})^*)(g) &= \int_K \alpha_k(\tilde{a} * (\tilde{b})^* (k^{-1}gk))\, dk \\ &= \int_K \alpha_k(a\alpha_k^{-1} gk(b^*))\, dk \\ &= \int_K \alpha_k(a)\, \alpha_{gk}(b^*)\, dk. \end{aligned}$$

We now use the fact that $a, b \in A_1(\pi)$. Therefore

$$a = P^\alpha(\pi)(a) = \sum_i P^\alpha_{ii}(\pi)(a)$$

and

$$b^* = P^\alpha(\bar{\pi})(b^*) = \sum_j P^\alpha_{jj}(\bar{\pi})(b^*).$$

For simplicity, during this proof we shall denote $P^\alpha_{ij}(\pi)(a) = a_{ij}$ and $P^\alpha_{ij}(\bar{\pi})(b^*) = b^*_{ij}$. With these notations, we have

$$P(\tilde{a} * (\tilde{b}))(g) = \int \alpha_k(a)\, \alpha_{gk}(b^*)\, dk = \sum_i \sum_j \int \alpha_k(a_{ii})\, \alpha_{gk}(b^*_{jj})\, dk.$$

Using Remark 2.1(iii) we further have

$$\begin{aligned} P(\tilde{a} * (\tilde{b})^*)(g) &= \sum_{i,j,r,l} \left(\int \pi_{ri}(k)\, \bar{\pi}_{lj}(gk)\, dk \right) a_{ir} b^*_{jl} \\ &= \sum_{i,j,r,l,s} \bar{\pi}_{ls}(g) \left(\int \pi_{ri}(k)\, \bar{\pi}_{sj}(k)\, dk \right) a_{ir} b^*_{jl}. \end{aligned}$$

Taking into account the orthogonality relations we have

$$
\begin{aligned}
P(\tilde{a} * (\tilde{b})^*)(g) &= \frac{1}{d(\pi)} \sum_{i,l,s} \bar{\pi}_{ls}(g)\, a_{is} b_{il}^* \\
&= \frac{1}{d(\pi)} \sum_{i,s} a_{is} \left(\sum \bar{\pi}_{ls}(g)\, b_{il}^* \right) \\
&= \frac{1}{d(\pi)} \sum_{i,s} a_{is} \alpha_g(b_{is}^*) \\
&= \frac{1}{d(\pi)} \sum_{i,s} c_{si} \alpha_g(b_{is}^*) \\
&= \frac{1}{d(\pi)} \operatorname{tr}([c_{ij}][b_{ij}^*](1 \times \bar{\pi}_g)) \qquad \text{(by Remark 2.1(iv))} \\
&= \frac{1}{d(\pi)^2} \psi([c_{ij}][b_{ij}^*]),
\end{aligned}
$$

where $c_{ij} = a_{ji}$ and the matrix $[c_{ij}]$ belongs to $A_2(\bar{\pi})$ and $[b_{ij}^*]$ belongs to $A_2(\bar{\pi})$. Therefore $P(S_{\pi,\iota} * \bar{S}_{\iota,\pi}) \subset \psi(\overline{A_2((\bar{\pi}) * A_2(\bar{\pi})})$. The reverse inclusion is obtained by reversing the arguments.

3. Saturated Actions and Simple Crossed Products

Let (A, G, α) be a C^*-dynamical system. There are several approaches to questions concerning the ideal structure of the crossed products $G \times_\alpha A$.

We shall review them following [9].

The first involves the Effros–Hahn conjecture for amenable G [10]. The second, in the context of abelian groups of automorphisms, involves the Connes spectrum [16, 17].

The third involves the notion of outer automorphism groups [8] and the fourth the notion of partly inner automorphism [22].

In particular, Olesen and Pedersen [16] proved that if G is discrete, then $G \times_\alpha A$ is simple if and only if (a) the Connes spectrum $\Gamma(\alpha) = \hat{G}$ and (b) A is G-simple (that is, A contains no G-invariant two-sided ideals).

This result is false if G is not discrete. Bratteli [2] has given a counterexample for $G = T$ (the circle group) and A the UHF algebra of type 2^∞. In an appendix to Bratteli's paper, Rosenberg [25] has "explained" somewhat Bratteli's counterexample. He proved that if G is compact and $G \times_\alpha A$ is simple then the fixed point algebra A^α must be simple. Corollary 3.8 shows that if G is compact abelian then $G \times_\alpha A$ is simple if and only if (a) the Arveson spectrum $\operatorname{Sp}(\alpha) = \hat{G}$ and (b) A_α is simple.

For a different approach to the above problem for G compact we send to [19]. In this section we shall consider a compact subgroup $K \subset G$ and give conditions for simplicity $G \times_\alpha A$ in terms of the subspaces S_{π_1, π_2} introduced above.

First, we shall exhibit a left-$S_{\pi,\iota}$–right-$G \times_\alpha A$ bimodule X which in some cases will be imprimitivity bimodule in the sense of [20].

Let $X = (G \times_\alpha A) * \chi_\iota$. Then, the convolution turns X into a left-S_ι–right-$G \times_\alpha A$ bimodule. We define the following S_ι-valued (respectively $G \times_\alpha A$-valued) inner products on X:

$$\langle x, y \rangle_{S_\iota} = y^* * x$$

$$(\text{respectively } \langle x, y \rangle_{G \times_\alpha A} = x * y^*).$$

It is then a matter of standard computations in $G \times_\alpha A$ that X is a left S_ι-rigged space in the sense of [20, Definition 2.8] and the conditions (1)–(3) of [20, Definition 6.10] are satisfied. Thus X may fail to be an S_ι–$G \times_\alpha A$ imprimitivity bimodule in the sense of [20, Definition 6.10] only in that the range of the $G \times_\alpha A$-valued inner product need not be dense.

This can be fixed by making the following definition, which for the case $G = K$ has been given by Rieffel (see [19, Chap. 7]).

3.1. Definition. Let (A, G, α) and K be as above. We say that α is K-saturated if X is an S_ι–$G \times_\alpha A$ imprimitivity bimodule. By reformulating the denseness of the range of the $G \times_\alpha A$-valued inner product we obtain:

3.2. Lemma. *The action α is K-saturated if and only if the two-sided ideal generated by S_ι coincides with $G \times_\alpha A$ (i.e., S_ι is full in $G \times_\alpha A$).*

Proof. Plainly the range of the $G \times_\alpha A$-valued inner product is the closed two-sided ideal $(G \times_\alpha A) * (G \times_\alpha A)$ generated by χ_ι and this last ideal coincides with the closed two-sided ideal generated by S_ι. Therefore $(G \times_\alpha A) * \chi_\iota * (G \times_\alpha A) = G \times_\alpha A$ if and only if S_ι is full.

3.3. Theorem. *Let (A, G, α) be a C^*-dynamical system and $K \subset G$ a compact subgroup. Then α is K-saturated if and only if $S_{\pi,\iota} * \overline{S_{\iota,\pi}} = S_\pi$ for all $\pi \in \hat{K}$.*

Proof. Assume that α is saturated and let $\pi \in \hat{K}$. By Lemma 3.2, $(G \times_\alpha A) * \chi_\iota * (G \times_\alpha A) = G \times_\alpha A$. By multiplying the above equality on both sides with χ_π we obtain

$$\chi_\pi * (G \times_\alpha A) * \chi_\iota * (G \times_\alpha A) * \chi_\pi = \chi_\pi * (G \underset{\alpha}{\times} A) \chi_\pi.$$

Hence $\overline{S_{\pi,\iota} * S_{\iota,\pi}} = S_\pi$. Conversely assume that $S_{\pi,\iota} * \overline{S_{\iota,\pi}} = S_\pi$ for all $\pi \in K$. We shall prove that the ideal $(G \times_\alpha A) * \chi_\iota * (G \times_\alpha A)$ is dense in $G \times_\alpha A$. By

Lemma 2.5(iii), $\sum_{\pi \in \hat{K}} \chi_\pi * (G \times A)$ is dense in $G\alpha A$. Therefore, in order to prove that $(G \times_\alpha A) * \chi_\iota * (G \times_\alpha A)$ is dense, it is sufficient to prove that every element in $\sum_\pi \chi_\pi * (G \times_\alpha A)$ can be approximated (in norm) by elements of $(G \times_\alpha A) * \chi_\iota * (G \times_\alpha A)$.

Let $\pi \in \hat{K}$ and $f \in G \times A$ such that $f = \chi_\pi * f \neq 0$. Since $S_{\pi,\iota} * S_{\iota,\pi}$ is a dense ideal of S_π, then there exists an approximate identity (e_λ) of S_π contained in that ideal. We claim that (norm) $\lim e_\lambda f = f$. Indeed

$$\begin{aligned}\|e_\lambda * f - f\|^2 &= \|(e_\lambda * f - f) * (f^* * e_\lambda - f*)\| \\ &= \|e_\lambda * f * f^* * e_\lambda - e_\lambda * f * f^* - f * f^* * e_\lambda + f * f^*\|.\end{aligned}$$

Taking into account that $f * f^* \in S_\pi$ and (e_λ) is an approximate identity of S_π, the claim follows. Since $e_\lambda \in S_{\pi,l} * S_{\iota,\pi}$, it immediately follows that $e_\lambda * f \in (G \times_\alpha A) * \chi_\iota * (G \times_\alpha A)$. So this ideal is dense and α is K-saturated.

We can now prove our result about simplicity of crossed products.

3.4. THEOREM. *Let (A, G, α) be a C^*-dynamical system and $K \subset G$ a compact subgroup. Then the following conditions are equivalent:*

(i) *$G \times_\alpha A$ is simple and*

(ii) (a) *$S_{\pi,\iota} \neq (0)$ for all $\pi \in \hat{K}$ and* (b) *I_π is simple for all $\pi \in \hat{K}$ (equivalently, S_π is simple for all $\pi \in \hat{K}$).*

In this case $G \times_\alpha A$ is strongly Morita equivalent with S_π, $\pi \in \hat{G}$.

Proof. Assume (i). Suppose by contradiction that $S_{\pi,\iota} = (0)$ for some $\pi \in \hat{K}$. Then $S_{\pi,\iota} = \chi_\pi * \overline{(G \times_\alpha A) * \chi_\iota} = (0)$. From this, it follows that the ideal $(G \times_\alpha A) * \chi_\iota * (G \times_\alpha A)$ is proper and therefore $G \times_\alpha A$ is not simple. Hence (i) implies (ii)(a).

We now prove (ii)(b). Since $G \times_\alpha A$ is simple, then by [5, Proposition 5.1], every hereditary subalgebra of $G \times_\alpha A$, in particular S_π, is simple. By Corollary 2.8(i) this is equivalent with I_π being simple.

Conversely assume (ii). Then because S_π are simple and $S_{\pi,\iota} * S_{\iota,\pi}$ are two-sided ideals of S_π, it follows that $\overline{S_{\pi,\iota} * S_{\iota,\pi}} = S_\pi$. By Theorem 3.3, $G \times_\alpha A$ is strongly Morita equivalent with S_ι, which is simple by hypothesis. Applying [21, Theorem 3.1] it follows that $G \times_\alpha A$ is simple. The last part of the theorem follows from the (easily verified) fact that $S_{\pi,\iota}$ is an S_π–S_ι imprimitivity bimodule.

Now, we shall specialize the above notions and give equivalent formulations of the preceding results in the case $G = K$. In this case, if $f \in C(G, A)$, then $(f * \chi_\iota)(g) = \int_K F(k)\, dk = a \in A$ (for some $\tilde{a}$). Therefore $f * \chi_\iota = a$ for some $a \in A^k$. It follows that the bimodule X is the closure of $\{\tilde{a} \mid a \in A\}$. In the notation of [19], $X = \bar{A}$. On the other hand, in this case

(G compact) S_ι may be identified with A^α [25]. It is then easy to see that the above left-S_ι-right-$G\times_\alpha A$ bimodule structure of X coincides with that defined by Rieffel [19, Chap. 7]. The following is a consequence of Theorem 3.3:

3.5. COROLLARY. *Let (A, G, α) be a C^*-dynamical system with G compact. Then α is saturated if and only if $\overline{A_2(\pi) * A_2(\pi)} = A \otimes B(H_\pi)^{\alpha\otimes ad\pi}$ for all $\pi \in \hat{G}$.*

Proof. From Lemmas 2.10 and 2.9 it follows that $\overline{A_2(\pi) * A_2(\pi)} = A \otimes B(H_\pi)^{\alpha\otimes ad\pi}$ if and only if $S_{\pi,\iota} * \overline{S_{\iota,\pi}} = S_\pi$. The corollary follows from Theorem 3.3.

If G is compact abelian, then for each $\pi \in \hat{G}$ we have

$$A_1(\pi) = A_2(\pi) = \{a \in A \mid \alpha_g(a) = \langle g, \pi\rangle a\}.$$

The following result is due to Rieffel.

3.6. COROLLARY [19, Theorem 7.1.15]. *Let (A, G, α) be a C^*-dynamical system with G compact abelian. Then α is saturated if and only if $A_\pi * \overline{A_\pi} = A^\alpha$ for all $\pi \in \hat{G}$. Further, we shall derive consequences of Theorem 3.4 for compact groups.*

3.7. COROLLARY. *Let (A, G, α) be a C^*-dynamical system with G compact. The following conditions are equivalent:*

(i) *$G \times_\alpha A$ is simple and*

(ii) (a) *$A_1(\pi) \neq (0)$ for all $\pi \in \hat{G}$ and* (b) *$A \otimes B(H_\pi)^{\alpha\otimes ad\pi}$ is simple for all $\pi \in \hat{G}$.*

Proof. From Lemma 2.6(v) it follows that $A_1(\pi) \neq (0)$ if and only if $S_{\pi,\iota} \neq (0)$. On the other hand, by Lemma 2.9, $A \otimes B(H_\pi)^{\alpha\otimes ad\pi}$ is $*$-isomorphic with I_π. Therefore $A \otimes B(H_\pi)^{\alpha\otimes ad\pi}$ is simple for all $\pi \in G$ if and only if I_π is simple for all $\pi \in \hat{G}$. It then follows that the condition (ii) of Corollary 3.7 is equivalent with the condition (ii) of Theorem 3.4 (for compact G) so the corollary follows from Theorem 3.4.

If G is compact abelian then the set $\{\pi \in G \mid A_1(\pi) \neq (0)\}$ is the Arveson spectrum $\mathrm{sp}(\alpha)$ [1, 18]. Therefore:

3.8. COROLLARY. *Let (A, G, α) be a C^*-dynamical system with G compact abelian. The following conditions are equivalent:*

(i) *$G \times_\alpha A$ is simple and*

(ii) (a) $\mathrm{sp}(\alpha) = \hat{G}$ *and* (b) *A^α is simple.*

Proof. Obviously, if G is abelian we have $A \otimes B(H_\pi)^{\alpha \otimes ad\pi} = A^\alpha$ for all $\pi \in \hat{G}$. The corollary follows from the preceding one.

If G is compact not necessarily abelian for $\pi = \iota$, we obviously have $A \otimes B(H_\iota)^{\alpha \otimes ad_\iota} = A^\alpha$. One may ask whether in Corollary 3.7 the simplicity of A^α together with the condition (ii)(a) still implies the simplicity of the crossed product. The answer is no as shown by the following.

3.9. EXAMPLE. Let $G = S_3$ be the permutation group on three elements. Then G has order 6 and is generated by two elements r, s satisfying the relations $r^3 = e$, $s^2 = e$, $rs = sr^2$. The dual $\hat{G}$ of G consists of (the classes of) three representations π_1, π_2, π_3 with $d(\pi_1) = d(\pi_2) = 1$ and $d(\pi_3) = 2$, where

$$\pi_1(r) = 1, \qquad \pi_1(s) = 1$$

$$\pi_2(r) = 1, \qquad \pi_2(s) = -1$$

$$\pi_3(r) = \begin{bmatrix} -1/2 & \sqrt{3/2} \\ -\sqrt{3/2} & -1/2 \end{bmatrix}, \qquad \pi_3(s) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since π_3 is the only element in $\hat{G}$ with $d(\pi_3) = 2$, we have $\pi_3 = \overline{\pi_3}$. A simple application of [26, Corollary 2, 11.1] shows that $\pi_3 = \pi_1 + \pi_2 + \pi_3$.

Let $A = M_2$ be the algebra of 2×2 matrices and define $\alpha = ad\pi_3$. Then α is ergodic on A so $A^\alpha = C \cdot 1$ is simple. Since α is equivalent with $\pi_3 \otimes \overline{\pi_3}$ we have that $A_1(\pi) \neq (0)$ for all $\pi \in G$. However, $A \otimes B(H_{\pi_3})^{\alpha \otimes ad\pi_3} = (M_2 \otimes M_2)^{ad\pi_3 \otimes ad\pi_3}$ is not simple because its centre is not trivial.

We consider now the similar question for prime crossed products.

Recall that a C^*-algebra A is called prime if for all nonzero two-sided ideals J_1, J_2 of A, their product is not zero, or equivalently if for every two (positive) elements a, b in the multiplier algebra $M(A)$ of A, we have $a \cdot A \cdot b \neq (0)$.

For actions of an abelian group on a W^*-algebra A, the crossed product is a factor if and only if the action is ergodic on the center of A and the Connes spectrum $\Gamma(\alpha)$ equals G [3, 4].

Research into the A^* analog of this result was started in [13, 15] with the study of compact abelian actions on prime or simple C^*-algebras. A result, proved independently in [13, Theorem 4.3; 15, Theorem 1], is that if A is prime and G is compact abelian then A^α is prime if and only if $\mathrm{Sp}(\alpha) = \Gamma(\alpha)$. One of our results (Corollary 3.12) is that if G is compact abelian then $G \times_\alpha A$ is prime if and only if A^α is prime and $\mathrm{sp}(\alpha) = \hat{G}$. This is an alternate characterization of primeness of $G \times_\alpha A$ (for compact abelian G) of that given by Olesen and Pedersen [16] for every locally compact abelian G.

3.10. *Remarks.* (i) If $B \subset A$ is a hereditary C^*-subalgebra of A then $\overline{BAB} = B$.

(ii) If A is a prime C^*-algebra then every hereditary C^*-subalgebra B of A is prime.

(iii) Let A be a C^*-algebra. If A contains an essential ideal J that is also a prime C^*-algebra then A is prime.

Proof. Part (i) follows from Effros' characterization of hereditary C^*-subalgebras [7].

(ii) Assume by contradiction that B is not prime. Then there exist two non-zero positive elements a, $b \in B$ such that $aBb = (0)$. By (i) above $aA \subset B$. Therefore $a^2Ab^2 = a(aAb)\, b \subset aBb = (0)$, which contradicts the fact that A is prime.

(iii) is straightforward.

3.11. THEOREM. *Let (A, G, α) be a C^*-dynamical system and $K \subset G$ a compact subgroup. The following conditions are equivalent:*

(i) *$G \times_\alpha A$ is prime and*

(ii) (a) *$S_{\pi,\iota} \neq (0)$ for all $\pi \in \hat{K}$ and* (b) *S_π are prime for all $\pi \in \hat{K}$.*

Proof. Assume that $G \times_\alpha A$ is prime. Since χ_π, χ_ι are non-zero elements of $M(G \times_\alpha A)$ we have $S_{\pi,\iota} = \chi_\pi * (G \times_\alpha A) * \chi_\iota \neq (0)$. On the other hand, S_π are hereditary C^*-subalgebras of the prime algebra $G \times_\alpha A$, so by Remark 3.10(ii) they are prime. Conversely, assume (ii), and let $I = (G \times_\alpha A) * \chi_\iota * (G \times_\alpha A)$ be the two-sided ideal of $G \times_\alpha Ag$ generated by χ_ι. We claim that I is an essential ideal of $G \times_\alpha A$. In order to prove the claim it is enough to prove that $f * I \neq (0)$ for all $f \in G \times_\alpha A$, $f \neq 0$. Let $f \in G \times_\alpha A$, $f \neq 0$. By Lemma 2.5(ii) there exists $\pi \in K$ such that $f * \chi_\pi \neq 0$. Then $\chi_\pi * f^* * f * \chi_\pi \neq 0$. Now, $\chi_\pi * I * \chi_\pi = \chi_\pi * (G \times_\alpha A) * \chi_\iota * (G \times_\alpha A) * \chi_\pi = S_{\pi,\iota} * S_{\iota,\pi}$. Therefore $\chi_\pi * I * \chi_\pi \neq (0)$ by (ii)(b). On the other hand, $S_{\pi,\iota} * S_{\iota,\pi}$ (and hence $\chi_\pi * I * \chi_\pi$) is a two-sided ideal of S_π by Lemma 2.6(iv). Since S_π is prime by our hypothesis (ii)(b), it follows that $\chi_\pi * f^* * f * \chi_\pi * I * \chi_\pi \neq (0)$. From this, it follows that $f * \chi_\pi * I \neq (0)$, so $f * I \neq (0)$. Therefore I is an essential ideal.

We now claim that I is a prime C^*-algebra. Indeed, by [20, Example 6.7], I is strongly Morita equivalent with $\chi_\iota * (G \times_\alpha A) * \chi_\iota$, which is prime by assumption. By [21, Theorem 3.1], I is prime. Therefore $G \times_\alpha A$ contains an essential ideal I which is prime. By Remark 3.10(ii), $G \times_\alpha A$ is prime.

Using arguments similar to the ones used in the proofs of Corollaries 3.7 and 3.8 we obtain the following consequences of the above theorem:

3.12. COROLLARY. *Let (A, G, α) be a C^*-dynamical system with G compact. Then the following conditions are equivalent:*

(i) $G \times_\alpha A$ *is prime and*

(ii) (a) $A_1(\pi) \neq (0)$ *for all* $\pi \in \hat{G}$ *and* (b) $A \otimes B(H_\pi)^{\alpha \otimes ad\pi}$ *is prime for all* $\pi \in \hat{G}$.

3.13. COROLLARY. *Let* (A, G, α) *be a* C^**-dynamical system with* G *compact abelian. Then the following conditions are equivalent*:

(i) $G \times_\alpha A$ *is prime and*

(ii) (a) $\mathrm{Sp}(\alpha) = \hat{G}$ *and* (b) A^α *is prime.*

Example 3.9 shows that if G is compact (even finite) not necessarily abelian, the primeness of A^α together with the condition (ii)(a) of Corollary 4.5 does not necessarily imply that $G \times_\alpha A$ is prime.

An easy application of the above results shows that the corresponding result for von Neumann algebras also holds.

ACKNOWLEDGMENTS

I am indebted to H. Halpern, V. Kaftal, and G. Weiss for many useful discussions and to Chris Phillips for a copy of his thesis [19]. This work was partially supported by the NSF under Grant DMS-8502353.

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